ON WAYS OF INTRODUCING A SMALL PARAMETER INTO EQUATIONS OF NONLINEAR VIBRATIONS

(O SPOSOBE VVEDENIIA MALOGO PARAMETRA V URAVNENIIA NELINEINYKH KOLEBANII)

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A.P. PROSKURIAKOV
(Moscow)

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The method of a small parameter has been widely used in the theory of nonlinear vibrations. The construction of the periodic solution by this method consists in systematic determination of the coefficients of a series in powers of a small parameter. After formal construction of such a series, there arises the question of estimating the radius of convergence. Little has yet been done in this field.

The problem of the convergence of the series which represents the periodic solution of the system is closely related to the manner in which the small parameter is introduced into the vibrating system. However, the investigator cannot always use the method of the small parameter. In some instances the small parameter is determined by the nature of the problem. Nevertheless we are interested in the various ways in which the small parameter may be introduced.

Consider the system of two nonlinear differential equations of the first order. We will show the results obtained by the introduction of the small parameter in various ways. We have*

$$\frac{dx}{dt} = -y + xf(r^2), \quad \frac{dy}{dt} = x + yf(r^2) \tag{1}$$

The function $f(r^2)$ is represented by an n-polynomial in powers of $r^2 = x^2 + y^2$

$$f(r^2) = b_n r^{2n} + \ldots + b_1 r^2 + b_0$$

Multiplying the first equation of (1) with x and the second equation with y, and then rearranging the terms, we have

* This example was suggested by N.G. Chetaev.

$$\frac{1}{2}\frac{dr^2}{dt} = r^2 f(r^2)$$

It is evident that this equation has the periodic solution r^2 = const, which represents the roots of the right-hand side of the equation. From these solutions we choose those which satisfy the initial condition

$$y(0) = 0 \tag{2}$$

We obtain $x = 0$, $y = 0$, and

$$x = A^* \cos t, \quad y = A^* \sin t \tag{3}$$

where A^* represents the roots of the equation $f(A^{*2}) = 0$.

The system (1) does not possess any other periodic solutions.

We introduce the small parameter μ in the ordinary way,

$$\frac{dx}{dt} = -y + \mu x f(r^2), \qquad \frac{dy}{dt} = x + \mu y f(r^2) \tag{4}$$

The solution of equation (4) is desired in the form of a series in powers of μ:

$$x(t) = x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \ldots$$
$$y(t) = y_0(t) + \mu y_1(t) + \mu^2 y_2(t) + \ldots$$

which has to satisfy the initial conditions (2). For $\mu = 0$ we have the solution

$$x_0 = A\cos t, \quad y_0 = A\sin t$$

which depends on the arbitrary parameter A. For the determination of x_1 and y_1 we obtain the equations

$$\frac{dx_1}{dt} = -y_1 + Af(A^2)\cos t, \qquad \frac{dy_1}{dt} = x_1 + Af(A^2)\sin t$$

These equations give the periodic solution for $Af(A^2) = 0$.

It is not difficult to establish by further calculations that all x_n and y_n for n = 1, 2, ... are equal to zero. Thus, the system (4) has the same solution as the system (1). We obtain the family of solutions, depending on the parameter μ; hence in this way the parameter which is introduced is not eliminated.

Note that Van der Pol's method leads to the previous result, as was shown [1], since his method represents only the first approximation in the method of small parameters.

We introduce the small parameter μ in another way, namely

$$\frac{dx}{dt} = \mu[-y + xf(r^2)], \quad \frac{dy}{dt} = \mu[x + yf(r^2)] \tag{5}$$

This system for $\mu = 0$, using the initial conditions (2), has the solutions $x_0 = a_0$, $y_0 = 0$.

From the conditions of periodicity of the functions x_1 and y_1 it follows that $a_0 = 0$. Further calculation shows that all x_n and y_n are equal to zero. Thus, by introducing the parameter μ in this manner, not only are we unable to obtain the family of the periodic solutions depending on this parameter, but also we are unable to find the periodic solution of equation (3), which belongs to the system (5) for $\mu = 1$. In this case, it follows that no other solution of the system can be found from the zero solution of the generating system.

By introducing the small parameter in the above manner, it is not possible to obtain one parameter family of periodic solutions, depending on the parameter. However, Chetaev has shown that by using another way of introduction, this can be done. We take the function $f(r^2)$ in the form

$$f(r^2) = r^2 (r^2 - A_1^2)$$

We introduce the parameter μ as follows

$$\frac{dx}{dt} = -y + xr^2 (r^2 - \mu^2 A_1^2), \qquad \frac{dy}{dt} = x + yr^2 (r^2 - \mu^2 A_1^2) \tag{6}$$

For $\mu = 0$ we have the solution $x_0 = 0$, $y_0 = 0$.

Further we obtain

$$x_n = a_n \cos t, \qquad y_n = a_n \sin t \qquad (n = 1, 2, 3, 4)$$

For the determination of x_5 and y_5 we have the equations

$$\frac{dx_5}{dt} = -y_5 + a_1^3 (a_1^2 - A_1^2) \cos t, \qquad \frac{dy_5}{dt} = x_5 + a_1^3 (a_1^2 - A_1^2) \sin t$$

From the condition of the periodicity of the functions x_5 and y_5 it follows that $a_1 = A_1$ or $a_1 = 0$.

Further calculations show that all x_n and y_n $(n = 2, 3, \ldots.)$ are equal to zero. Consequently, in addition to the zero solution, the system (6) also has a periodic solution depending on the parameter

$$x = \mu A_1 \cos t, \qquad y = \mu A_1 \sin t$$

Note that the result of equation (6) is not changed if we introduce the parameter μ into the coefficient in such a way that the second parts of the right-hand side of both equations (6) have the same coefficient. Only the calculations will be changed.

Thus, by successful introduction of the small parameter, we succeed in obtaining periodic solutions even when the generating equation has a zero solution.

Omitting the calculations, we give a Table in which the different ways of introducing the parameter μ into the system (1) are given. The results of several versions are also included in the Table. In all versions the zero solution is omitted.

Version IV is similar to version III, but has in addition an isolated periodic solution. In version V, to obtain the solution of the generating system, we use the periodic solution $x_0 = A_1 \cos t$, $y_0 = A_1 \sin t$, with a prescribed value of A_1. In version VI the family, which represents the periodic solutions, has a finite radius of convergence.

We notice that in the given example the introduction of a small parameter is not suitable, in using the transformation of independent variables with indeterminate constants (Liapunov), because the period of the period of the solution of the system of equations under consideration appears as a constant quantity, equal to 2π.

From the examples considered above it can be concluded that other versions can be obtained. The periodic solutions of the system can be obtained, if the solution of the generating system forms a family of solutions dependent on the arbitrary parameter. As an example of such a system we can mention quasilinear systems.

If the solution of the generating system appears as an isolated periodic solution, or even as a zero solution, then the parameter μ must be introduced in such a way that it is possible to construct a family of periodic solutions which depend on this parameter, having a large enough radius of convergence of the series representing this family of solutions.

BIBLIOGRAPHY

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TABLE I.

Type of variation	System of Equations	Periodic solutions using the method of small parameter
I	$\dot{x} = -y + \mu x f(r^2)$ $\dot{y} = x + \mu y f(r^2)$	$x = A^* \cos t,\ y = A^* \sin t$ $f(A^{*2}) = 0$
II	$\dot{x} = \mu[-y + x f(r^2)]$ $\dot{y} = \mu[x + y f(r^2)]$	Unobtainable
III	$\dot{x} = -y + x r^2(r^2 - \mu^2 A_1^2)$ $\dot{y} = x + y r^2(r^2 - \mu^2 A_1^2)$	$x = \mu A_1 \cos t,\quad y = \mu A_1 \sin t$
IV	$\dot{x} = -y + x(r^2 - \mu^2)(r^2 - A_1^2)$ $\dot{y} = x + y(r^2 - \mu^2)(r^2 - A_1^2)$	$x = \mu \cos t,\quad y = \mu \sin t$ $x = A_1 \cos t,\ y = A_1 \sin t$
V	$\dot{x} = -y + x r^2[r^2 - (A_1 + \mu)^2]$ $\dot{y} = x + y r^2[r^2 - (A_1 + \mu)^2]$	$x = (A_1 + \mu)\cos t,\ y = (A_1 + \mu)\sin t$
VI	$\dot{x} = -y + \frac{x r^2}{c^2}[(c - \mu)^2 r^2 - c^2 A_1^2]$ $\dot{y} = x + \frac{y r^2}{c^2}[(c - \mu)^2 r^2 - c^2 A_1^2]$ $c > 0$	$x = \frac{A_1}{1 - \mu / c} \cos t$ $y = \frac{A_1}{1 - \mu / c} \sin t$ $\mu < c$